

RIGHT-ANGLED ARTIN GROUPS ON FINITE SUBGRAPHS OF DISK GRAPHS

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ABSTRACT. Koberda proved that if a graph Γ is a full subgraph of a curve graph $\mathcal{C}(S)$ of an orientable surface S , then the right-angled Artin group $A(\Gamma)$ on Γ is a subgroup of the mapping class group $\text{Mod}(S)$ of S . On the other hand, for a sufficiently complicated surface S , Kim-Koberda gave a graph Γ which is not contained in $\mathcal{C}(S)$, but $A(\Gamma)$ is a subgroup of $\text{Mod}(S)$. In this paper, we prove that if Γ is a full subgraph of a disk graph $\mathcal{D}(H)$ of a handlebody H , then $A(\Gamma)$ is a subgroup of the handlebody group $\text{Mod}(H)$ of H . Further, we show that there is a graph Γ which is not contained in some disk graphs, but $A(\Gamma)$ is a subgroup of the corresponding handlebody groups.

1. INTRODUCTION

Let $H = H_{g,n}$ be an orientable 3-dimensional handlebody of genus g with n marked points. We regard its boundary ∂H as a compact connected orientable surface $S = S_{g,n}$ of genus g with n marked points. We denote by

$$\xi(H) = \max\{3g - 3 + n, 0\}$$

the *complexity* of H , a measure which coincides with the number of components of a maximal multi-disk in H . We also define the complexity $\xi(S)$ of S as $\xi(S) = \max\{3g - 3 + n, 0\}$. Let Γ be a finite simplicial graph. Through this paper, we denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ respectively. The *right-angled Artin group* on Γ is defined by

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

For two groups G_1 and G_2 , we write $G_1 \leq G_2$ if there is an embedding from G_1 to G_2 , that is, an injective homomorphism from G_1 to G_2 . Similarly, we write $\Lambda \leq \Gamma$ for two graphs Γ and Λ if Λ is isomorphic to an induced subgraph of Γ . We denote by $\text{Mod}(H)$ and $\text{Mod}(S)$ the handlebody group of H and the mapping class group of S respectively.

Right-angled Artin groups were introduced by Baudisch [1]. Recently, these groups have attracted much interest from 3-dimensional topology and geometric group theory through the work of Haglund-Wise [6],[7] on special cube complexes. In particular, various mathematicians investigate subgroups of right-angled Artin groups or right-angled Artin subgroups of groups. Crisp-Sageev-Sapir [5] studied surface subgroups of right-angled Artin groups. Kim-Koberda [12] proved that for any tree T , there exists a pure braid group PB_n such that $A(T)$ is embedded in PB_n . Bridson [3] proved that the isomorphism problem for the mapping class group of a surface whose genus is sufficiently large is unsolvable by using right-angled Artin

subgroup in mapping class groups. See Koberda [14] for other researches about right-angled Artin groups and their subgroups.

On the other hand, the geometry of mapping class groups of surfaces is well understood. A handlebody group $\text{Mod}(H)$ of H is a subgroup of the mapping class group $\text{Mod}(S)$ of S . Hamenstädt-Hensel [8] showed that $\text{Mod}(H)$ is exponentially distorted in $\text{Mod}(S)$. Therefore, the geometric properties of handlebody groups may be different from those of mapping class groups. Furthermore, disk graphs are not quasi-isometric to curve graphs (see Masur-Schleimer [15]). Our motivation of this article is whether the following three propositions are true when we change the assumptions of mapping class groups and curve graphs to handlebody groups and disk graphs.

Proposition 1.1. ([13, Theorem 1.1 and Proposition 7.16]) *If $\Gamma \leq \mathcal{C}(S)$, then $A(\Gamma) \leq \text{Mod}(S)$.*

Proposition 1.2. ([11, Theorem 2]) *Let S be an orientable surface with $\xi(S) \leq 2$. If $A(\Gamma) \leq \text{Mod}(S)$, then $\Gamma \leq \mathcal{C}(S)$.*

Proposition 1.3. ([11, Theorem 3]) *Let S be an orientable surface with $\xi(S) \geq 4$. Then there exists a finite graph Γ such that $A(\Gamma) \leq \text{Mod}(S)$ but $\Gamma \not\leq \mathcal{C}(S)$.*

Definition 1.4. An embedding f from $A(\Gamma)$ to $\text{Mod}(H)$ is *standard* if f satisfies the following two conditions.

- (i) The map f maps each vertex of Γ to a multi-disk twist;
- (ii) For two distinct vertices u and v of Γ , the support of $f(u)$ is not contained in the support of $f(v)$.

We first prove the following three theorems.

Theorem 1.5. *If $\Gamma \leq \mathcal{D}(H)$, then $A(\Gamma) \leq \text{Mod}(H)$.*

Theorem 1.6. *Let H be a handlebody with $\xi(H) = 0$ or $\xi(H) = 1$. If $A(\Gamma) \leq \text{Mod}(H)$, then $\Gamma \leq \mathcal{D}(H)$. Let H be a handlebody with $\xi(H) = 2$. If there exists a standard embedding $f: A(\Gamma) \rightarrow \text{Mod}(H)$, then $\Gamma \leq \mathcal{D}(H)$.*

Theorem 1.7. *For $H = H_{0,7}$ and $H = H_{1,5}$, there exists a finite graph Γ such that $A(\Gamma) \leq \text{Mod}(H)$ but $\Gamma \not\leq \mathcal{D}(H)$.*

From Theorem 1.7, it follows that the converse of Theorem 1.5 is generally not true. Further, Kim-Koberda proved that having N -thick stars forces the converse of Proposition 1.1.

Proposition 1.8. ([11, Theorem 5]) *Suppose S is a surface with $\xi(S) = N$ and Γ is a finite graph with N -thick stars. If $A(\Gamma) \leq \text{Mod}(S)$, then $\Gamma \leq \mathcal{C}(S)$.*

We also prove the following:

Theorem 1.9. *Suppose H is a handlebody with $\xi(H) = N$ and Γ is a finite graph with N -thick stars. If there is a standard embedding $f: A(\Gamma) \rightarrow \text{Mod}(H)$, then $\Gamma \leq \mathcal{D}(H)$.*

Note that our all theorems also hold when we change handlebody groups to pure handlebody groups. We also note that we cannot apply the argument of Kim-Koberda [11] for handlebody groups of high complexity handlebodies. The methods in this paper are worthless for high complexity handlebodies.

Problem 1.10. *When is the converse of Theorem 1.5 true?*

2. PRELIMINARIES

2.1. Graph-theoretic terminology. In this paper, a *graph* is a one-dimensional simplicial complex. In particular, graphs have neither loops nor multi-edges. For $X \subseteq V(\Gamma)$, the *subgraph of Γ induced by X* is the subgraph Λ of Γ defined by $V(\Lambda) = X$ and

$$E(\Lambda) = \{e \in E(\Gamma) \mid \text{the end points of } e \text{ are in } X\}.$$

In this case, we also say Λ is an *induced subgraph* or a *full subgraph* of Γ . A graph Γ is Λ -*free* if no induced subgraphs of Γ are isomorphic to Λ . In particular, Γ is *triangle-free* if no induced subgraphs of Γ are triangles. The *link* of v in Γ is the set of the vertices in Γ which are adjacent to v , and denoted as $\text{Link}(v)$. The *star* of v is the union of $\text{Link}(v)$ and $\{v\}$, and denoted as $\text{St}(v)$. A *clique* is a subset of the vertex set which spans a complete subgraph. By a link, a star, or a clique, we often also mean the subgraphs induced by them. For a positive integer N , we say Γ has *N -thick stars* if each vertex v of Γ is contained in two cliques $K_1 \cong K_2$ on N vertices of Γ whose intersection is exactly v . Equivalently, $\text{Link}(v)$ contains two disjoint copies of complete graphs on $N - 1$ vertices of Γ for each vertex v .

2.2. Handlebodies. A *handlebody* H_g of genus g is a compact orientable 3-dimensional manifold constructed by attaching g one-handles $D^2 \times I$ to a 3-ball, where D^2 is a 2-disk and I is an interval. The boundary ∂H_g of H_g is a closed connected orientable surface S_g of genus g . A handlebody $H = H_{g,n}$ of genus g with n marked points is a handlebody of genus g , together with n pairwise distinct points p_1, p_2, \dots, p_n on ∂H_g . We regard the boundary ∂H of H as a compact connected orientable surface $S = S_{g,n}$ of genus g with n marked points. A disk d is *properly embedded* in H if its boundary ∂d is embedded in ∂H , and its interior is embedded in the interior of H . A properly embedded disk d is *essential* if the simple closed curve ∂d is essential in ∂H , that is, ∂d does not bound a disk in ∂H or is not isotopic to a marked point on ∂H . By a *disk* in H we mean a properly embedded essential disk $(d, \partial d) \subseteq (H, \partial H)$. A *disk twist* δ_d along a disk d in H is the homeomorphism defined by cutting H along d , twisting one of the sides by 2π to the right, and gluing two sides of d back to each other (see Figure 1).

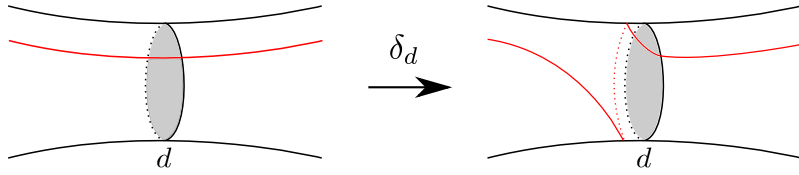


FIGURE 1. A disk twist δ_d along a disk d in H .

A *multi-disk* in H is the union of a finite collection of disjoint disks in H . The number of components of a multi-disk is at most $3g - 3 + n$. A multi-disk is *maximal* if the number of its components is $3g - 3 + n$.

The *handlebody group* $\text{Mod}(H)$ of H is the group of orientation preserving homeomorphisms of H , fixing the marked points setwise, up to ambient isotopy. The *mapping class group* $\text{Mod}(S)$ of S is the group defined by changing the role of homeomorphisms of H into homeomorphisms of S in the definition of the handlebody group. The *pure handlebody group* $\text{PMod}(H)$ of H is the group of orientation

preserving homeomorphisms of H , fixing the marked points pointwise, up to ambient isotopy. We note that it is not important to distinguish between handlebody groups and pure handlebody groups in our considerations, since our all theorems hold for both handlebody groups and pure handlebody groups. We call elements of $\text{Mod}(H)$ or $\text{Mod}(S)$ mapping classes. An element Φ of $\text{Mod}(H)$ is a *multi-disk twist* if Φ can be represented by a composition of powers of disk twists along disjoint pairwise-non-isotopic disks.

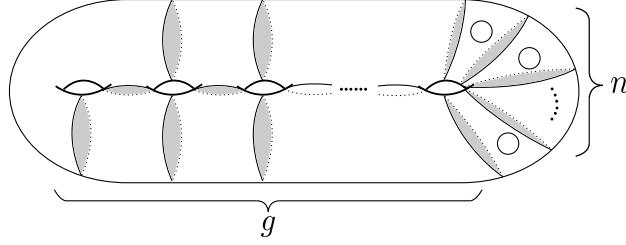


FIGURE 2. An example of a maximal multi-disk in $H_{g,n}$.

An element Φ of $\text{Mod}(S)$ is *pseudo-Anosov* if $\Phi^n(\alpha) \neq \alpha$ for any isotopy class α of simple closed curve on S and $n \geq 1$ (see [2]). An element Φ of $\text{Mod}(H)$ is *pseudo-Anosov* if its restriction $\Phi|_{\partial H}$ to ∂H is a pseudo-Anosov element of $\text{Mod}(\partial H) = \text{Mod}(S)$. The *support* $\text{supp}(\phi)$ of a homeomorphism ϕ of H is defined by

$$\text{supp}(\phi) = \overline{\{p \in H \mid \phi(p) \neq p\}}.$$

Similarly, we define the support of a homeomorphism ϕ of S as

$$\text{supp}(\phi) = \overline{\{p \in S \mid \phi(p) \neq p\}}.$$

The *disk graph* $\mathcal{D}(H)$ of H is a graph whose vertex set is the set of isotopy classes of disks in H . Two vertices are adjacent if the corresponding isotopy classes admit disjoint representatives. The *curve graph* $\mathcal{C}(S)$ of S is a graph defined by changing the role of disks in the definition of the disk graph into properly embedded essential simple closed curves in S . By a *curve* in S we mean a properly embedded essential simple closed curve in S . There exists a natural inclusion $\mathcal{D}(H) \rightarrow \mathcal{C}(S)$ given by sending an isotopy class of a disk d to the isotopy class of a curve ∂d . Slightly abusing the notation, we often realize isotopy classes of disks or curves as disks or curves.

3. PROOF OF THEOREM 1.5

To prove Theorem 1.5, it is sufficient to show the following lemma.

Lemma 3.1. *Let Γ be a finite graph and H a handlebody. Let i be an embedding from Γ to $\mathcal{D}(H)$ as an induced subgraph. Then for all sufficiently large N , the map*

$$i_{*,N}: A(\Gamma) \rightarrow \text{Mod}(H)$$

given by sending v to the N th power $\delta_{i(v)}^N$ of a disk twist $\delta_{i(v)}$ along $i(v)$ is injective.

We use the following lemma to prove Lemma 3.1.

Lemma 3.2. ([11, Theorem 7 (1)]) *Let Γ be a finite graph and S an orientable surface. Let i' be an embedding from Γ to $\mathcal{C}(S)$ as an induced subgraph. Then for all sufficiently large N , the map*

$$i'_{*,N}: A(\Gamma) \rightarrow \text{Mod}(S)$$

given by sending v to the N th power $T_{i'(v)}^N$ of a Dehn twist $T_{i'(v)}$ along $i'(v)$ in S is injective.

Proof of Lemma 3.2. For the proof, see [13]. \square

Proof of Lemma 3.1. Let $i: \Gamma \rightarrow \mathcal{D}(H)$ be an embedding as an induced subgraph. Recall that $\mathcal{D}(H)$ is a subgraph of $\mathcal{C}(S)$, and so there is a natural embedding $j: \mathcal{D}(H) \rightarrow \mathcal{C}(S)$ given by sending a disk d to the boundary circle ∂d . We set $i' = j \circ i$. Then i' is an embedding of Γ into $\mathcal{C}(S)$ as an induced subgraph. By Lemma 3.2, there is a sufficiently large $N > 0$ such that the map $i'_{*,N}: A(\Gamma) \rightarrow \text{Mod}(S)$ given by sending $v \in V(\Gamma)$ to $T_{i'(v)}^N$ is injective. Since $i'(v) = j \circ i(v) = \partial(i(v))$, a Dehn twist $T_{i'(v)}$ along $i'(v)$ is extended to a disk twist $\delta_{i(v)}$ along $i(v)$ in H . Therefore, the map $i_{*,N}: A(\Gamma) \rightarrow \text{Mod}(H)$ given by sending $v \in V(\Gamma)$ to $\delta_{i(v)}^N$ is injective. \square

From Lemma 3.1, $A(\Gamma)$ is a subgroup of $\text{Mod}(H)$, and we have finished a proof of Theorem 1.5.

4. PROOF OF THEOREM 1.6

The idea of the proof of Theorem 1.6 comes from the proof of [11, Theorem 3] by changing the assumptions of mapping class groups and curve graphs to handlebody groups and disk graphs. However, for $H = H_{1,0}$ and $H = H_{1,1}$ we can not apply their argument and we prove it by another way. First we remark the following.

Remark 4.1. In Definition 1.4, if $\text{supp}(f)$ is a maximal clique in $\mathcal{D}(H)$, then the condition (ii) implies that v is an isolated vertex.

Proof of Theorem 1.6. First we consider the case $\xi(H) = 0$, that is, $g = 0$ and $n \leq 3$, or $g = 1$ and $n = 0$. If $g = 1$ and $n = 0$, then there exists one disk in $H_{1,0}$. Thus, this case comes down to the case $\xi(H) = 1$. We may assume that $g = 0$ and $n \leq 3$. Then the handlebody groups are trivial. Note that there is no essential disk in H . We assume that $A(\Gamma)$ is a subgroup of $\text{Mod}(H)$. Then $A(\Gamma)$ is also trivial. Therefore $A(\Gamma)$ has no generator, and so Γ has no vertex. Hence $\Gamma \leq \mathcal{D}(H)$.

Suppose that $\xi(H) = 1$, that is, $H = H_{0,4}$, $H = H_{1,0}$, or $H = H_{1,1}$. First, we assume $H = H_{0,4}$. Note that $\mathcal{D}(H)$ is an infinite union of isolated vertices. $\text{Mod}(H)$ is virtually free, since $\text{Mod}(S)$ is virtually free and subgroups of virtually free groups are also virtually free. We assume that $A(\Gamma)$ is a subgroup of $\text{Mod}(H)$. Then $A(\Gamma)$ is free because it is virtually free and torsion-free. Hence, $A(\Gamma)$ has no relation, and so Γ is a graph consists of finite isolated vertices. Therefore $\Gamma \leq \mathcal{D}(H)$. Secondly, we assume $H = H_{1,0}$, or $H = H_{1,1}$. We note that there is only one essential disk in H (see Hamenstädt-Hensel [8, Section 2]). We call the disk d . Hence, $\mathcal{D}(H)$ is a graph consists of a single vertex. On the other hand, $\text{Mod}(H) \cong \langle \delta_d \rangle \cong \mathbb{Z}$, where $\langle \delta_d \rangle$ is the group generated by a disk twist δ_d along d (see Hamenstädt-Hensel [8, Proposition 2.2]). We assume that $A(\Gamma)$ is a subgroup of $\text{Mod}(H)$. Then, $A(\Gamma)$ is trivial or \mathbb{Z} because any non-trivial subgroup of \mathbb{Z} is isomorphic to \mathbb{Z} . If $A(\Gamma)$ is trivial, then Γ has no vertex. Hence $\Gamma \leq \mathcal{D}(H)$. If $A(\Gamma)$ is isomorphic to \mathbb{Z} , then Γ is a graph consists of a single vertex. Therefore $\Gamma \leq \mathcal{D}(H)$.

Suppose that $\xi(H) = 2$, that is, $H = H_{0,5}$ or $H = H_{1,2}$. We note that $\mathcal{D}(H)$ is triangle-free. First, we claim that the conclusion of the theorem holds for Γ if and only if it holds for each connected component of Γ . This is an easy consequence of the fact that $\mathcal{D}(H)$ has infinite diameter and that there exists a pseudo-Anosov homeomorphism on H . So, we may assume that Γ is connected, and so it has at least one edge. By the hypothesis, there is a standard embedding f from $A(\Gamma)$ to $\text{Mod}(H)$. Each vertex v of $A(\Gamma)$ is mapped to a power of a single disk twist δ_d along d by f , since Γ has no isolated vertex and $\mathcal{D}(H)$ is triangle-free (see Remark 4.1). Hence we gain an embedding $\Gamma \rightarrow \mathcal{D}(H)$. \square

5. PROOF OF THEOREM 1.7

Let Γ_0 and Γ_1 be the finite graphs shown in Figure 3. We denote by C_4 the 4-cycle spanned by $\{a, b, c, d\}$.

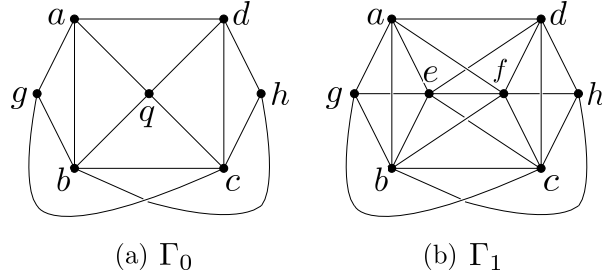


FIGURE 3. Two graphs Γ_0 and Γ_1 .

Let $\phi: A(\Gamma_0) \rightarrow A(\Gamma_1)$ be the map defined by $\phi(q) = ef$ and $\phi(v) = v$ for any $v \in V(\Gamma_0) - \{q\}$. For a graph Γ , we will denote by $\langle v \rangle$ the subgroup of $A(\Gamma)$ generated by $v \in V(\Gamma)$.

Lemma 5.1. ([11, Lemma 11]) *The map $\phi: A(\Gamma_0) \rightarrow A(\Gamma_1)$ is injective.*

Proof. For the proof see the proof of Lemma [11, Lemma 11]. \square

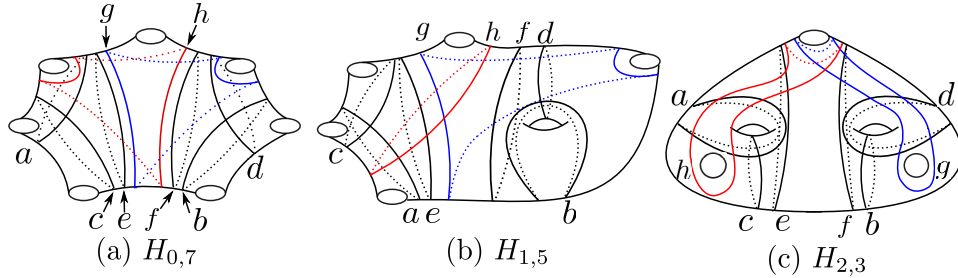


FIGURE 4. Handlebodies $H_{0,7}$, $H_{1,5}$, and $H_{2,3}$.

Lemma 5.2. *The graph Γ_1 is embedded into $\mathcal{D}(H)$ if and only if H is a handlebody with $\xi(H) \geq 6$, $H = H_{0,7}$, or $H = H_{1,5}$.*

We introduce the intersection number between two disks d_1 and d_2 in H as follows.

$$i(d_1, d_2) = |\partial d_1 \cap \partial d_2|.$$

Two disks d_1 and d_2 in H are in *minimal position* if the intersection number is minimal in the isotopy classes of d_1 and d_2 . Note that two disks in H intersect if and only if the boundary circles intersect in ∂H . We also remark that two curves are in minimal position in S if and only if they do not bound any bigons on S .

Proof of Lemma 5.2. We show Γ_1 is embedded into $\mathcal{D}(H_{0,7})$, $\mathcal{D}(H_{1,5})$, and $\mathcal{D}(H_{2,3})$ as an induced subgraph. Note that the complexities of $H_{0,7}$, $H_{1,5}$, and $H_{2,3}$ are four, five, and six respectively. We put disks a, b, c, d, e, f, g , and h in the handlebodies as in Figure 4 so that they form the graph Γ_1 in the disk graphs. One can verify that the disks are in minimal position, since their boundary circles do not bound bigons on the boundary surfaces. Hence Γ_1 is embedded into $\mathcal{D}(H_{0,7})$, $\mathcal{D}(H_{1,5})$, and $\mathcal{D}(H_{2,3})$ as an induced subgraph. Therefore Γ_1 is also embedded into a disk graph of a handlebody whose complexity is at least six. We can also show that Γ_1 is not embedded into any other disk graphs (see Section 7 for the proof). We have thus proved the lemma. \square

Let H be a handlebody with $\xi(H) = 4$ or $\xi(H) = 5$. Suppose $\{a, b, c, d\}$ are disks in H which form a four cycle C_4 in $\mathcal{D}(H)$ with this order. Let S_1 be a regular neighborhood of ∂a and ∂c in ∂H , and S_2 a regular neighborhood of ∂b and ∂d in ∂H so that $S_1 \cap S_2 = \emptyset$. Set $S_0 = \overline{\partial H - (S_1 \cup S_2)}$. Note that we regard the boundaries of S_0 , S_1 , and S_2 as marked points from now.

Lemma 5.3. *Let H be a handlebody with $\xi(H) = 5$. Then, the triple (S_0, S_1, S_2) satisfies exactly one of the following seventeen cases, possibly after switching the roles of S_1 and S_2 .*

- (1) $(S_1, S_2) \in \{(S_{0,4}, S_{0,6}), (S_{0,4}, S_{1,3}), (S_{0,5}, S_{0,5}), (S_{0,5}, S_{1,2}), (S_{1,2}, S_{1,2})\}$, $S_0 \approx S_{0,2}$, and S_0 intersects both S_1 and S_2 .
- (2) $(S_1, S_2) \in \{(S_{0,4}, S_{0,5}), (S_{0,4}, S_{1,2})\}$, $S_0 \approx S_{0,3}$, and S_0 intersects each of S_1 and S_2 at only one boundary component.
- (3) $S_1 \approx S_{0,4}$, $S_2 \in \{S_{0,5}, S_{1,2}\}$, $S_0 \approx S_{0,2} \amalg S_{0,2}$, and each component of S_0 intersects both S_1 and S_2 .
- (4) $S_1 \approx S_{0,4}$, $S_2 \in \{S_{0,5}, S_{1,2}\}$, $S_0 \approx S_{0,2} \amalg S_{0,2}$, and one component of S_0 intersects each of S_1 and S_2 at only one boundary component, while the other component of S_0 intersects S_1 at just two boundary components.
- (5) $S_1 \approx S_{0,4}$, $S_2 \in \{S_{0,5}, S_{1,2}\}$, $S_0 \approx S_{0,2} \amalg S_{0,3}$ such that the $S_{0,2}$ component intersects both S_1 and S_2 and the $S_{0,3}$ component is disjoint from S_2 , and moreover, $S_{0,3} \cap S_1 \approx S^1$.
- (6) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,4}$, and S_0 intersects each of S_1 and S_2 at only one boundary component.
- (7) $S_1, S_2 \approx S_{0,4}$, $S_0 \in \{S_{0,2} \amalg S_{0,4}, S_{0,2} \amalg S_{1,1}\}$ such that the $S_{0,2}$ component intersects both S_1 and S_2 and the $S_{0,4}$ (resp. $S_{1,1}$) component is disjoint from S_2 , and moreover, $S_{0,4} \cap S_1 \approx S^1$ (resp. $S_{1,1} \cap S_1 \approx S^1$).
- (8) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,3}$, and $S_0 \cap S_1 \approx S^1 \amalg S^1$ and $S_0 \cap S_2 \approx S^1$.
- (9) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,3}$, and both components of S_0 intersects each of S_1 and S_2 at only one boundary component respectively.

- (10) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,3}$ such that the $S_{0,2}$ component intersects both S_1 and S_2 and the $S_{0,3}$ component is disjoint from S_2 , and moreover, $S_{0,3} \cap S_1 \approx S^1$.
- (11) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,3}$ such that the $S_{0,3}$ component intersects both S_1 and S_2 and the $S_{0,2}$ component is disjoint from S_2 , and moreover, $S_{0,2} \cap S_1 \approx S^1 \amalg S^1$.
- (12) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,2}$, and three components of S_0 intersect both S_1 and S_2 .
- (13) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,2}$, and two components of S_0 intersect both S_1 and S_2 , while the other component of S_0 is disjoint from S_2 and intersects S_1 at just two boundary components.
- (14) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,2}$ such that one component of S_0 intersects both S_1 and S_2 , one (named I) of the other components is disjoint from S_2 , the other component (named J) is disjoint from S_1 , and $I \cap S_1 \approx S^1 \amalg S^1$ and $J \cap S_2 \approx S^1 \amalg S^1$.
- (15) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,3}$ such that two $S_{0,2}$ components intersect both S_1 and S_2 and the $S_{0,3}$ component is disjoint from S_2 , and moreover, $S_{0,3} \cap S_1 \approx S^1$.
- (16) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,3}$, such that one $S_{0,2}$ component intersects both S_1 and S_2 , the other $S_{0,2}$ component (named I) is disjoint from S_2 , the $S_{0,3}$ component is disjoint from S_1 or S_2 (here we suppose that $S_{0,3}$ is disjoint from S_2), and moreover, $I \cap S_1 \approx S^1 \amalg S^1$ and $S_{0,3} \cap S_1 \approx S^1$.
- (17) $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,3} \amalg S_{0,3}$ such that the $S_{0,2}$ component intersects both S_1 and S_2 and each $S_{0,3}$ component is disjoint from S_1 or S_2 respectively (here we suppose that both $S_{0,3}$ components are disjoint from S_2), and moreover, each $S_{0,3}$ component intersects S_1 at only one boundary component.

Proof of Lemma 5.3. Let α be the number of free isotopy classes of boundary components of S_0 that are contained in $S_1 \cup S_2$. We have $\alpha > 0$, since S is connected and $S_1 \cap S_2 = \emptyset$. Let $\xi(S_0)$ be the sum of complexities of the components of S_0 . Then $\xi(\partial H) = \xi(S_1) + \xi(S_2) + \xi(S_0) + \alpha$. Since S_i ($i = 1, 2$) contains at least one curve, we have $2 \leq \xi(S_1) + \xi(S_2)$. Further, $\xi(S_1) + \xi(S_2) = 5 - \xi(S_0) - \alpha \leq 5 - 0 - 1 = 4$. Therefore it follows that $2 \leq \xi(S_1) + \xi(S_2) \leq 4$. We note that if S_1 or S_2 is a surface whose complexity is one, then it is homeomorphic to only $S_{0,4}$. In fact, if it is homeomorphic to $S_{1,1}$, then it cannot have two curves ∂a and ∂c since $H_{1,1}$ has only one isotopy class of disk.

We suppose that $\xi(S_1) + \xi(S_2) = 4$. Then we have $\xi(S_0) + \alpha = 1$. If $\xi(S_1) = 1$ and $\xi(S_2) = 3$, then $S_1 \approx S_{0,4}$ and $S_2 \approx S_{0,6}, S_{1,3}$. If $\xi(S_1) = 2$ and $\xi(S_2) = 2$, then $S_1 \approx S_{0,5}, S_{1,2}$ and $S_2 \approx S_{0,5}, S_{1,2}$. By the assumption that $\alpha \geq 1$, we have $\alpha = 1$ and $\xi(S_0) = 0$. Since S_0 has at least two boundary components, $S_0 \approx S_{0,2}$ or $S_0 \approx S_{0,3}$. If $S_0 \approx S_{0,3}$, then this contradicts the assumption that $\alpha = 1$. Hence, we have $S_0 \approx S_{0,2}$. Case (1) is immediate.

We suppose that $\xi(S_1) + \xi(S_2) = 3$. Then we have $\xi(S_0) + \alpha = 2$. Without loss of generality we may assume $\xi(S_1) = 1$ and $\xi(S_2) = 2$. It follows that $S_1 \approx S_{0,4}$ and $S_2 \approx S_{0,5}, S_{1,2}$. If $\alpha = 1$, then S_0 forced to be an annulus and we have a contradiction of the fact that $\xi(S_0) + \alpha = 2$. So we have $\alpha = 2$ and $\xi(S_0) = 0$. If S_0 is connected, then $\alpha = 2$ implies that $S_0 \approx S_{0,3}$, and hence Case (2) follows. We assume that S_0 is not connected. By the assumption that $\alpha = 2$, the number

of connected components of S_0 is at most two. We have $S_0 \approx S_{0,2} \amalg S_{0,2}$ or $S_0 \approx S_{0,2} \amalg S_{0,3}$. If $S_0 \approx S_{0,2} \amalg S_{0,2}$, then we have two cases where each component intersects both S_1 and S_2 , and one (we name it I) of the component is disjoint from S_2 . In the former case, we obtain Case (3). In the latter case, if $S_1 \cap I \approx S^1$, then there is no essential disk in I , and so we have a contradiction to $\alpha = 2$. Hence Case (4) follows. If $S_0 \approx S_{0,2} \amalg S_{0,3}$, then $S_{0,3}$ has to be disjoint from S_2 and $S_{0,3} \cap S_1 \approx S^1$ since $\alpha = 2$. Then we obtain Case (5).

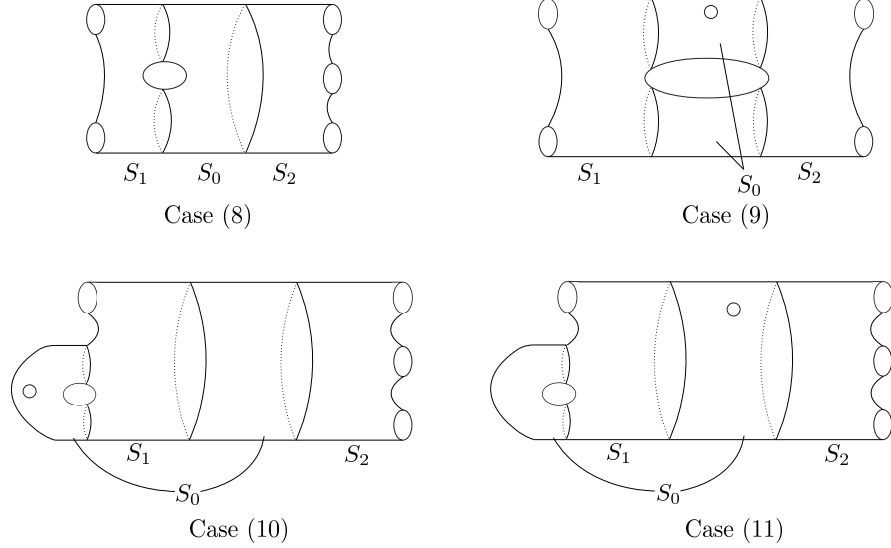


FIGURE 5. The handlebody of Cases (8), (9), (10), and (11).

We suppose that $\xi(S_1) + \xi(S_2) = 2$. Then $\xi(S_1) = 1$ and $\xi(S_2) = 1$. Thus $S_1 \approx S_{0,4}$ and $S_2 \approx S_{0,4}$. Moreover it follows that $\xi(S_0) + \alpha = 3$. If $\alpha = 1$, then by a similar argument to that of Case (1) we have $S_0 \approx S_{0,2}$, and this contradicts the fact that $\xi(S_0) = 2$. First, we consider the case where $\alpha = 2$ and $\xi(S_0) = 1$. If S_0 is connected, then $\alpha = 2$ implies that $S_0 \approx S_{0,4}$ and $S_{0,4} \cap S_1 \approx S^1$, $S_{0,4} \cap S_2 \approx S^1$, hence Case (6) follows. We assume that S_0 is not connected. By the assumption that $\alpha = 2$, the number of connected components of S_0 is at most two and the component which intersect both S_1 and S_2 has to be $S_{0,2}$. The other component (we name I) which is disjoint from S_2 is $S_{0,4}$ or $S_{1,1}$, and $I \cap S_1 \approx S^1$. Then Case (7) follows. Next, we consider the case where $\alpha = 3$ and $\xi(S_0) = 0$. If S_0 is connected, then $\alpha = 3$ implies that $S_0 \approx S_{0,3}$, $S_{0,3} \cap S_1 \approx S^1 \amalg S^1$, and $S_{0,3} \cap S_2 \approx S^1$. Hence Case (8) follows (see Figure 5). We assume that S_0 is not connected. By the assumption that $\alpha = 3$, the number of connected components of S_0 is at most three. First we suppose that the number of connected components of S_0 is two. By the assumption that $\xi(S_0) = 0$, $S_0 \approx S_{0,2} \amalg S_{0,2}$ or $S_0 \approx S_{0,2} \amalg S_{0,3}$. If $S_0 \approx S_{0,2} \amalg S_{0,2}$, then this contradicts the assumption that $\alpha = 3$, and so we have $S_0 \approx S_{0,2} \amalg S_{0,3}$. If each component of S_0 intersects both S_1 and S_2 , then Case (9) is immediate (see Figure 5). If the $S_{0,2}$ component intersects both S_1 and S_2 , and $S_{0,3}$ component is disjoint from S_2 , then Case (10) is immediate (see Figure 5). If the $S_{0,3}$ component intersects both S_1 and S_2 , and $S_{0,2}$ component is disjoint from S_2 , then Case (11) is immediate (see Figure 5).

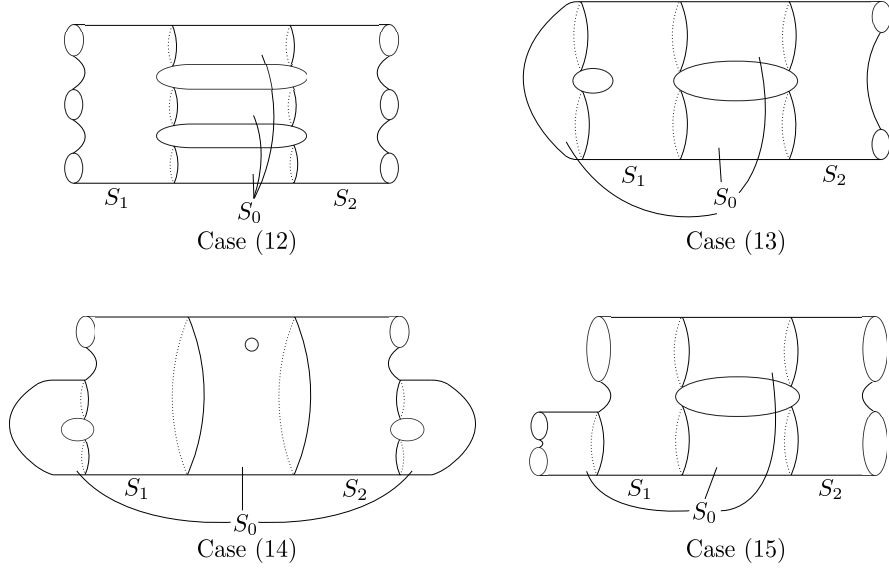


FIGURE 6. The handlebody of Cases (12), (13), (14), and (15).

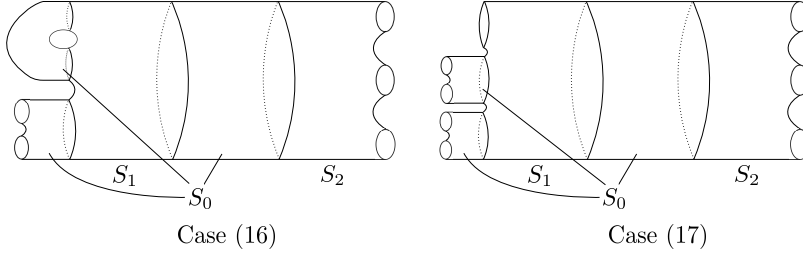


FIGURE 7. The handlebody of Cases (16) and (17).

Finally, we suppose that the number of connected components of S_0 is three. By the assumption that $\xi(S_0) = 0$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,2}$, $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,3}$, or $S_0 \approx S_{0,2} \amalg S_{0,3} \amalg S_{0,3}$. We note that if $S_0 \approx S_{0,3} \amalg S_{0,3} \amalg S_{0,3}$, then this contradicts the assumption that $\alpha = 3$. We assume that $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,2}$. If each component of S_0 intersects both S_1 and S_2 , then Case (12) is immediate (see Figure 6). If two components of S_0 intersect both S_1 and S_2 and the other component (we name it I) is disjoint from S_2 , then $I \cap S_1 \approx S^1 \amalg S^1$. We obtain Case (13) (see Figure 6). We assume that just one component of S_0 intersects both S_1 and S_2 . We also assume that one of the other component (we name it I) is disjoint from S_2 and the other component (we name it J) is disjoint from S_1 . Then $I \cap S_1 \approx S^1 \amalg S^1$ and $J \cap S_2 \approx S^1 \amalg S^1$, and so Case (14) follows (see Figure 6). We assume that $S_0 \approx S_{0,2} \amalg S_{0,2} \amalg S_{0,3}$. Note that the $S_{0,3}$ component of S_0 must not intersect both S_1 and S_2 since $\alpha = 3$. If each $S_{0,2}$ component intersects both S_1 and S_2 , then the $S_{0,3}$ component is disjoint from S_2 and $S_{0,3} \cap S_1 \approx S^1$, and Case (15) is immediate (see Figure 6). We assume that just one of the $S_{0,2}$ component intersects both S_1 and S_2 , and the other component (we name it I) is disjoint

from S_2 . We also suppose that the $S_{0,3}$ component is disjoint from S_2 . Then $I \cap S_1 \approx S^1 \amalg S^1$ and $S_{0,3} \cap S_1 \approx S^1$, hence Case (16) follows (see Figure 7). We assume that $S_0 \approx S_{0,2} \amalg S_{0,3} \amalg S_{0,3}$. Note that each of the $S_{0,3}$ components of S_0 must not intersect both S_1 and S_2 since $\alpha = 3$. Then the $S_{0,2}$ component intersects both S_1 and S_2 and two $S_{0,3}$ components are disjoint from S_2 , and moreover each $S_{0,3}$ component intersects S_1 at only one boundary component respectively. We obtain Case (17) (see Figure 7). \square

Lemma 5.4. *Let H be a handlebody with $\xi(H) = 4$. Then, the triple (S_0, S_1, S_2) satisfies exactly one of the following five cases, possibly after switching the roles of S_1 and S_2 .*

- (1)' $S_1 \in \{S_{1,2}, S_{0,5}\}$, $S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2}$, and S_0 intersects both S_1 and S_2 .
- (2)' $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,3}$, and S_0 intersects each of S_1 and S_2 at only one boundary component.
- (3)' $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2}$, and each component of S_0 intersects both S_1 and S_2 .
- (4)' $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,2}$, and one component of S_0 intersects each of S_1 and S_2 at only one boundary component, while the other component of S_0 intersects S_1 at two boundary components.
- (5)' $S_1, S_2 \approx S_{0,4}$, $S_0 \approx S_{0,2} \amalg S_{0,3}$ such that the $S_{0,2}$ component intersects both S_1 and S_2 and the $S_{0,3}$ component is disjoint from S_2 , and moreover, $S_{0,3} \cap S_1 \approx S^1$.

Note that we obtain Lemma 5.4 by changing the assumption of a surface S in [11, Lemma 13] to a handlebody H . We can prove this by the same process as the proof of Lemma 5.3. In our case, if S_1 and S_2 are surfaces whose complexities are one, then they are homeomorphic to only $S_{0,4}$. On the other hand, in [11, Lemma 13] if S_1 and S_2 are surfaces whose complexities are one, then they are homeomorphic to $S_{0,4}$ or $S_{1,1}$.

Lemma 5.5. *For $H = H_{1,5}$, there exists an embedding from $A(\Gamma_0)$ to $\text{Mod}(H)$, but Γ_0 does not embed into $\mathcal{D}(H)$ as an induced subgraph.*

Proof of Lemma 5.5. First, we show the first half of the conclusion. By Lemma 5.1, $A(\Gamma_0) \leq A(\Gamma_1)$. By Lemma 5.2 and Lemma 3.1, $A(\Gamma_1) \leq \text{Mod}(H)$. Therefore, it follows that $A(\Gamma_0) \leq \text{Mod}(H)$. For the second half, we suppose that $\Gamma_0 \leq \mathcal{D}(H)$. We regard a, b, c, d, g, h , and q as disks in H . From $C_4 \leq \Gamma_0$, we have one of the seventeen cases in Lemma 5.3. By the definition of disk graphs, the intersections $q \cap g$, $q \cap h$, $g \cap S_2$, $h \cap S_1$, and $g \cap h$ are not empty. Moreover $\partial q \subseteq S_0$ and $g \cap S_1 = h \cap S_2 = \emptyset$.

In Case (1), the annulus S_0 connects S_1 and S_2 . This implies that $\partial g \subseteq S_2$ and $\partial h \subseteq S_1$, and so $g \cap h = \emptyset$. This is a contradiction. In Case (2), since $\partial q \subseteq S_0 \approx S_{0,3}$, we have $\partial q = S_0 \cap S_1$ or $\partial q = S_0 \cap S_2$. Without loss of generality, we may assume $\partial q = S_0 \cap S_1$. The curve ∂q is a separating curve which separates S_1 from ∂H . By the fact that $g \cap S_1 = \emptyset$, we have $g \cap q = \emptyset$. This contradicts the fact that $g \cap q \neq \emptyset$. In Cases (3), (4), and (5), if ∂g intersects ∂h , then they intersect on the components of S_0 which connect S_1 and S_2 . Similarly to Case (1), we have $g \cap h = \emptyset$ and this is a contradiction. Case (6) does not appear for $H_{1,5}$. Note that we see Γ_0 is embedded in $\mathcal{D}(H_{0,8})$. In Case (7), by a similar argument to that of Case (1) we have $g \cap h = \emptyset$, and this is a contradiction. In Case (8), we have

$\partial q \subseteq S_0 \cap S_1$ or $\partial q = S_0 \cap S_2$ since $\partial q \subseteq S_0 \approx S_{0,3}$. Without loss of generality we may assume that $\partial q \subseteq S_0 \cap S_1$. By the fact that $g \cap q \neq \emptyset$, ∂g intersects ∂q . Then it follows that ∂g intersects S_1 . This is a contradiction. In Case (9), by a similar argument to that of Case (8), we have $g \cap q \neq \emptyset$. This is a contradiction. In Case (11), by a similar argument to that of Case (2), we have $g \cap q = \emptyset$. This contradicts the fact that $g \cap q \neq \emptyset$. In Cases (10), (12), (13), (14), (15), (16), and (17), by a similar argument to that of Case (3), we have $g \cap h = \emptyset$. This is a contradiction. Therefore it follows that $\Gamma_0 \not\leq \mathcal{D}(H_{1,5})$. \square

Lemma 5.6. ([11, Lemma 14]) *Let S be a surface with $\xi(S) = 4$. There exists an embedding from $A(\Gamma_0)$ to $\text{Mod}(S)$, but Γ_0 does not embed into $\mathcal{C}(S)$ as an induced subgraph.*

Proof of Lemma 5.6. For the proof see the proof of [11, Lemma 14]. \square

Lemma 5.7. *For $H = H_{0,7}$, there exists an embedding from $A(\Gamma_0)$ to $\text{Mod}(H)$, but Γ_0 does not embed into $\mathcal{D}(H)$ as an induced subgraph.*

Proof of Lemma 5.7. It directly follows from Lemma 5.6, since the handlebody group of H is isomorphic to the mapping class group of ∂H , and the disk graph of H is isomorphic to the curve graph of ∂H . \square

From Lemmas 5.5 and 5.7, we have completed the proof of Theorem 1.7. Therefore the converse of Theorem 1.5 is not true in general.

Remark 5.8. For a graph Γ , we define $\eta(\Gamma)$ to be the minimum of $\xi(H)$ among connected handlebodies H satisfying $\Gamma \leq \mathcal{D}(H)$. From Lemma 5.6, we see $\eta(\Gamma_0) > 4$. The graph Γ_0 embeds into $\mathcal{D}(H_{0,8})$ (see Figure 8, the handlebody of Case (6) in Lemma 5.3). Hence we see $\eta(\Gamma_0) = 5$. Further, there exists a handle body H whose complexity is four such that Γ_1 cannot embed into $\mathcal{D}(H)$. From these facts, we cannot apply the arguments of Kim-Koberda [11, Section 4.3] for high complexity cases to handlebodies. However, if H is a handlebody whose complexity is $n \geq 4$ and $H_{0,7} \subseteq H$, then there exists a graph Λ_n such that $A(\Lambda_n) \leq \text{Mod}(H)$ but $\Lambda_n \not\leq \mathcal{D}(H)$ by the same argument as that of the proof of [11, Proposition 16].

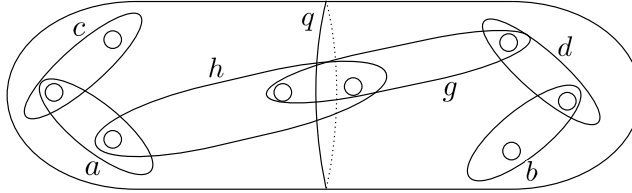


FIGURE 8. The graph Γ_0 is embedded into $\mathcal{D}(H_{0,8})$.

6. PROOF OF THEOREM 1.9

If A is a multi-disk on H , then we denote by $\langle A \rangle$ a subgroup of $\text{Mod}(H)$ which is generated by disk twists along the disks in A . The proof of Theorem 1.9 comes from the proof of Kim-Koberda [11, Theorem 5].

Proof of Theorem 1.9. Let v be an arbitrary vertex of $V(\Gamma)$. We write K and L for two disjoint cliques of Γ such that $K \amalg \{v\}$ and $L \amalg \{v\}$ are cliques on N vertices of Γ . We note that there are such cliques in Γ for any $v \in V(\Gamma)$ because of the assumption. The support of $f\langle K \rangle$ is a regular neighborhood of a multi-disk in H , and we call the multi-disk A . Similarly, we write B and C for multi-disks in the supports of $f\langle L \rangle$ and $f\langle v \rangle$ in H respectively. Since $\xi(H) = N$, multi-disks $A \cup C$ and $B \cup C$ are maximal. Note that $\langle C \rangle$ is a subgroup of $\langle A \cup C \rangle \cap \langle B \cup C \rangle$. By the diagram in Figure 9, $f\langle v \rangle$ is a finite index subgroup of $\langle C \rangle$. By the fact that $\langle C \rangle \cong \mathbb{Z}^{|C|}$ and $f\langle v \rangle \cong \mathbb{Z}$, it follows that $|C| = 1$. Hence, for each $v \in V(\Gamma)$, $f(v)$ is some single disk twist δ_d along a disk d , and we obtain an injection from Γ into $\mathcal{D}(H)$. \square

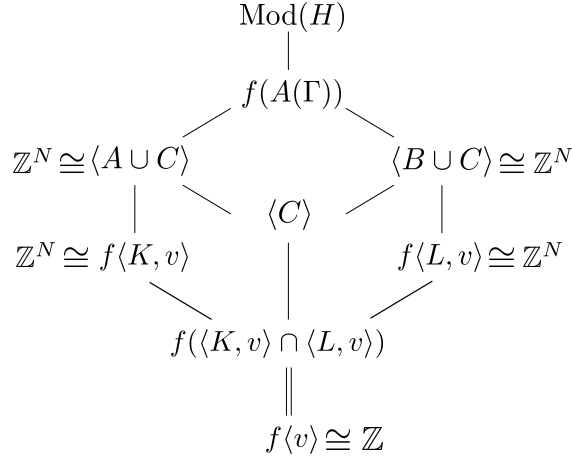


FIGURE 9. A figure representing a relationship between groups and their subgroups.

7. APPENDIX

In this section, we prove the following proposition.

Proposition 7.1. *The graph Γ_1 is not embedded into the disk graphs of H with $\xi(H) \leq 3$, $H = H_{1,4}$, $H = H_{2,1}$, and $H = H_{2,2}$.*

Proof of Proposition 7.1. First Γ_1 is not embedded into the disk graphs of H with $\xi(H) \leq 3$ because Γ_1 has cliques on four vertices. Next we suppose that H is a handlebody with $\xi(H) = 4$, that is, $H = H_{0,7}$, $H = H_{1,4}$, or $H = H_{2,1}$. We will show that Γ_1 is embedded into only $\mathcal{D}(H_{0,7})$. We suppose that $\Gamma_1 \leq \mathcal{D}(H)$. By the fact that $C_4 \leq \Gamma_1$, we obtain one of the five cases in Lemma 5.4. From the definition of disk graphs, we see $e \cap g = \emptyset$, $f \cap h = \emptyset$, $e \cap f = \emptyset$, $g \cap S_1 = \emptyset$, and $h \cap S_2 = \emptyset$. Further, $g \cap h \neq \emptyset$, $g \cap f \neq \emptyset$, $h \cap e \neq \emptyset$, $e \subseteq S_0$, and $f \subseteq S_0$. In Case (1)', the annulus S_0 connect S_1 and S_2 . This implies that $g \subseteq S_2$ and $h \subseteq S_1$, and so $g \cap h = \emptyset$. This is a contradiction. In Case (2)', S_1 and S_2 are homeomorphic to $S_{0,4}$. One can confirm that Γ_1 is embedded into only $\mathcal{D}(H_{0,7})$ in this case. Note that if we discuss it for a surface S with $\xi(S) = 4$, then S_1 and S_2 are homeomorphic to $S_{0,4}$ or $S_{1,1}$, and so Γ_1 is embedded into $\mathcal{C}(S_{0,7})$, $\mathcal{C}(S_{1,4})$, and

$\mathcal{C}(S_{2,1})$. In Cases (3)', (4)', and (5)', by the same argument as that of Case (1)', we have $g \cap h = \emptyset$, and this is a contradiction.

Secondly we suppose that H is a handlebody with $\xi(H) = 5$, that is, $H = H_{0,8}$, $H = H_{1,5}$, or $H = H_{2,2}$. We will show that Γ_1 is embedded into only $\mathcal{D}(H_{0,8})$ and $\mathcal{D}(H_{1,5})$. In Cases (1), (3), (4), (5), (7), (10), (11), (12), (13), (14), (15), (16), and (17), by the same argument that Γ_0 is not embedded into $\mathcal{D}(H_{1,5})$ in the proof of Lemma 5.5, we have $g \cap h = \emptyset$, and this is a contradiction. In Case (2), we see Γ_1 is embedded into only $\mathcal{D}(H_{0,8})$ and $\mathcal{D}(H_{1,5})$. In Case (6), we see Γ_1 is embedded into only $\mathcal{D}(H_{0,8})$. In Cases (8) and (9), we see Γ_1 is embedded into only $\mathcal{D}(H_{1,5})$. By the argument above, we have finished the proof of the proposition. \square

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